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Industrial & Systems
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The Incomplete Dirichlet's Multiple Integral

Research Report No. 76-24

by

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Abstract

The **incomplete Dirichlet's multiple integral** is defined as

$$\iint\cdots\int_R f(t_1 + t_2 + \cdots + t_n)\, t_1^{\alpha_1 - 1}\, t_2^{\alpha_2 - 1} \cdots t_n^{\alpha_n - 1}\, dt_1\, dt_2 \cdots dt_n$$

where $R = \{(t_1, t_2, \cdots, t_n):\ t_i \geq 0,\ \sum_1^n t_i \leq 1,\ t_{i+1} \leq a_i \sum_1^i t_j;\ i = 1, 2, \cdots, n-1\}$

It is shown that this integral is reducible to a single integral involving products of incomplete beta fuctions as a multiplicative constant. The results are generalized and extended to other multiple integrals. Finally, as an application a class of multivariate type distribution defined by a functional form is partially investigated.

1. Introduction

If $f(\cdot)$ is a continuous function of some real non-negative argument, and $\alpha_i > 0$, $i = 1, 2, \cdots, n$, then the Dirichlet's multiple integral

$$I_n = \underset{t_1+t_2+\cdots+t_n \leq 1}{\int\int\cdots\int} f(t_1 + t_2 + \cdots + t_n)\, t_1^{\alpha_1-1} t_2^{\alpha_2-1} \cdots t_n^{\alpha_n-1}\, dt\ dt_2 \cdots dt_n \qquad (1)$$

may be expressed as the single integral [5], [18]

$$I_n = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2) \cdots \Gamma(\alpha_n)}{\Gamma(\alpha_1 + a_2 + \cdots + \alpha_n)} \int_0^1 f(\tau)\tau^{(\alpha_1+\alpha_2+\cdots+\alpha_n)-1} d\tau$$

Alternatively, we may express the multiplicative quotient involving gamma functions in terms of products of beta functions and write

$$I_n = B(\alpha_2, \alpha_1)B(\alpha_3, \alpha_2 + \alpha_1) \cdots B(\alpha_n, \alpha_{n-1} + \alpha_{n-2} + \cdots + \alpha_2 + \alpha_1)$$
$$\int_0^1 f(\tau)\tau^{(\alpha_1+\alpha_2+\cdots+\alpha_n)-1} d\tau \qquad (2)$$

where in general

$$B(\ell, m) = \frac{\Gamma(\ell)\Gamma(m)}{\Gamma(\ell + m)}$$

An extension of (1) was proposed by Sivazlian [16] where in (1) the argument of the function $f(\cdot)$ was replaced by various partial sums of the variables t_1, t_2, $\cdots$, t_n. Further extensions in the integrand form of (1) were proposed by Sivazlian [17] and Klamkin [13]. Klamkin also suggested an interesting extension of (1) in yet another direction, namely by changing the region of integration to $\{(t_1, t_2, \ldots, t_n)\colon\ t_i \geq 0, \sum_1^r t_i \leq t, \sum_{r+1}^n t_i \leq t'\}$, $t, t' > 0$. The multiple integral is then reducible to a double integral again involving quotients of gamma functions as its multiplicative constant.

In the present work, we propose still another region R defined for $0 < a_i \leq \infty$, $i = 1, 2, \ldots, n$ as the simplex

$$R = \{(t_1, t_2, \ldots, t_n): t_i \geq 0, \sum_1^n t_i \leq 1, t_{i+1} \leq a_i \sum_1^i t_j; \quad i = 1, 2, \ldots, n-1\} \quad (3)$$

and define the <u>incomplete Dirichlet's multiple integral</u> as

$$I_n = \int\int_R \cdots \int f(t_1 + t_2 + \cdots + t_n)\, t_1^{\alpha_1 - 1} t_2^{\alpha_2 - 1} \ldots t_n^{\alpha_n - 1}\, dt_1\, dt_2 \ldots dt_n \quad (4)$$

This integral is <u>not</u> reducible to any of the previous forms investigated. However, in the very special case when $a_i = \infty$, the Liouville's extension of Dirichlet's integral, that is form (1) is recovered. In general, we shall show that the proposed multiple integral (4) has the same format as (2) except that the multiplicative constants outside the single integral are products of <u>incomplete beta functions</u>. We shall show that $\mathcal{I}_n = kI_n$, where k is the product of (n - 1) incomplete beta function ratio.

2. <u>The Incomplete Dirichlet's Multiple Integral</u>

To reduce (4), we first make the change in variable $t_i = x_i^2$, $i = 1, 2, \ldots, n$. The Jacobian of the transformation is clearly $2^n x_1 x_2 \ldots x_n$. The new region of integration $\hat{R}$ is defined by the set of points

$$\hat{R} \equiv \{(x_1, x_2, \ldots, x_n): x_i \geq 0, \sum_1^n x_i^2 \leq 1, x_{i+1} \leq \sqrt{a_i}\sqrt{x_1^2 + x_2^2 + \ldots + x_i^2} \text{ for } i = 1, 2, \ldots, n-1\}$$

Hence

$$\mathcal{I}_n = 2^n \int\int_{\hat{R}} \cdots \int f(x_1^2 + x_2^2 + \ldots + x_n^2)\, x_1^{2\alpha_1 - 1} x_2^{2\alpha_2 - 1} \cdots x_n^{2\alpha_n - 1}\, dx_1\, dx_2 \ldots dx_n$$

We now utilize a second change in variable which is the elegant transformation into generalized n-dimensional spherical coordinates proposed by Schläfli in 1855 [4], also by Clare in 1881 [5], and used often in statistics (e.g. see [12]). Let then

$$x_1 = r\cos\theta_{n-1}\cos\theta_{n-2}\cdots\cos\theta_3\cos\theta_2\cos\theta_1$$
$$x_2 = r\cos\theta_{n-1}\cos\theta_{n-2}\cdots\cos\theta_3\cos\theta_2\sin\theta_1$$
$$x_3 = r\cos\theta_{n-1}\cos\theta_{n-2}\cdots\cos\theta_3\sin\theta_2$$
$$\cdots$$
$$x_{n-1} = r\cos\theta_{n-1}\sin\theta_{n-2}$$
$$x_n = r\sin\theta_{n-1}$$

The Jacobian of the transformation is

$$r^{n-1}(\cos\theta_{n-1})^{n-2}(\cos\theta_{n-2})^{n-3}\cdots(\cos\theta_2)$$

To determine the new region of integration $\tilde{R}$, we note that the ball $0 \leq \sum_1^n x_i^2 \leq 1$ is mapped into the set of points $0 \leq r \leq 1$. Also, for $i = 1, 2, \ldots, n-1$, the set of points defined by

$$0 \leq x_{i+1} \leq \sqrt{a_i}\sqrt{x_1^2 + x_2^2 + \cdots + x_i^2}$$

is mapped into the set of points

$$0 \leq r\cos\theta_{n-1}\cos\theta_{n-2}\cdots\cos\theta_{i+2}\cos\theta_{i+1}\sin\theta_i$$
$$\leq \sqrt{a_i}\, r\cos\theta_{n-1}\cos\theta_{n-2}\cdots\cos\theta_{i+1}\cos\theta_i$$

or $\qquad 0 \leq \tan\theta_i \leq \sqrt{a_i}$

or $\qquad 0 \leq \theta_i \leq \tan^{-1}\sqrt{a_i}$

It thus follows that $\hat{R}$ is mapped into $\tilde{R}$ where

$$\tilde{R} = \{(r, \theta_1, \theta_2, \ldots, \theta_{n-1}):\ 0 \leq r \leq 1,\ 0 \leq \theta_i \leq \tan^{-1}\sqrt{a_i} \text{ for } i = 1, 2, \ldots, n-1\}$$

The expression for I can then be written as

$$\begin{aligned} I = 2^n \int\int\cdots\int_{\tilde{R}} f(r^2)\, & (r\cos\theta_{n-1}\cos\theta_{n-2}\cdots\cos\theta_2\cos\theta_1)^{2\alpha_1-1} \\ & (r\cos\theta_{n-1}\cos\theta_{n-2}\cdots\cos\theta_2\sin\theta_1)^{2\alpha_2-1} \\ & \cdots(r\cos\theta_{n-1}\sin\theta_{n-2})^{2\alpha_{n-1}-1}(r\sin\theta_{n-1})^{2\alpha_n-1} \\ & [r^{n-1}(\cos\theta_{n-1})^{n-2}(\cos\theta_{n-2})^{n-3}\cdots(\cos\theta_2)]\, dr\, d\theta_1\, d\theta_2\cdots d\theta_n \\ = 2^n\, [\int_0^1 & r^{2(\alpha_1+\alpha_2+\ldots+\alpha_n)-1} f(r^2)dr] \end{aligned}$$

$$[\int_0^{\tan^{-1}\sqrt{a_1}} (\sin\theta_1)^{2\alpha_2-1} (\cos\theta_1)^{2\alpha_1-1} d\theta_1]$$

$$[\int_0^{\tan^{-1}\sqrt{a_2}} (\sin\theta_2)^{2\alpha_3-1} (\cos\theta_2)^{2(\alpha_1+\alpha_2)-1} d\theta_2]$$

$$\cdots$$

$$[\int_0^{\tan^{-1}\sqrt{a_{n-2}}} (\sin\theta_{n-2})^{2\alpha_{n-1}-1} (\cos\theta_{n-2})^{2(\alpha_1+\alpha_2+\ldots+\alpha_{n-2})-1} d\theta_{n-2}]$$

$$[\int_0^{\tan^{-1}\sqrt{a_{n-1}}} (\sin\theta_{n-1})^{2\alpha_n-1} (\cos\theta_{n-1})^{2(\alpha_1+\alpha_2+\ldots+\alpha_{n-1})-1} d\theta_{n-1}]$$

Making the change in variable $\tau = r^2$ and $u_i = \sin^2\theta_i$, $i = 1, 2, \ldots, n - 1$, we obtain

$$I_n = [\int_0^{\frac{a_1}{1+a_1}} u_1^{\alpha_2-1} (1 - u_1)^{\alpha_1-1} du_1]\ [\int_0^{\frac{a_2}{1+a_2}} u_2^{\alpha_3-1} (1 - u_2)^{\alpha_2+\alpha_1-1} du_2]$$

$$\cdots\ [\int_0^{\frac{a_{n-1}}{1+a_{n-1}}} u_{n-1}^{\alpha_n-1} (1 - u_{n-1})^{\alpha_{n-1}+\alpha_{n-2}+\ldots+\alpha_2+\alpha_1-1} du_{n-1}]$$

$$[\int_0^1 f(\tau)\tau^{\alpha_1+\alpha_2+\cdots+\alpha_n-1} d\tau]$$

Using the definition of the incomplete beta function

$$B_x(\alpha, \beta) = \int_0^x y^{\alpha-1} (1 - y)^{\beta-1} dy$$

we obtain the following expression for I_n:

$$I_n = [B_{\frac{a_1}{1+a_1}}(\alpha_2, \alpha_1)][B_{\frac{a_2}{1+a_2}}(\alpha_3, \alpha_2 + \alpha_1)] \cdots [B_{\frac{a_{n-1}}{1+a_{n-1}}}(\alpha_n, \alpha_{n-1} + \alpha_{n-2} + \cdots + \alpha_1)]$$

$$\cdot \int_0^1 f(\tau)\tau^{\alpha_1+\alpha_2+\cdots+\alpha_n-1} d\tau \qquad (5)$$

Finally, if $I_x(a, b) = B_x(a, b)/B(a, b)$ is the incomplete beta function ratio, then clearly:

$$\mathcal{I}_n / I_n = \prod_{i=1}^{n-1} I_{\frac{a_i}{1+a_i}}(\alpha_i, \alpha_{i-1} + \alpha_{i-2} + \cdots + \alpha_1)$$

which is independent of the function $f(\cdot)$. In the special case when $a_i = \infty$ for all $i = 1, 2, \ldots, n - 1$, the value of the previous expression equals 1.

3. Some Generalizations

We consider two generalizations of particular interest although other classes of multiple integrals may be treated in the same fashion. First consider the multiple integral

$$\mathcal{I}_n(t) = \int\int_S \cdots \int f(u_1 + u_2 + \cdots + u_n) u_1^{\alpha_1 - 1} u_2^{\alpha_2 - 1} \cdots u_n^{\alpha_n - 1} du_1 du_2 \cdots du_n \tag{6}$$

where S is the simplex defined by the set of points

$$S = \{(u_1, u_2, \ldots, u_n) : u_i \geq 0, \ \sum_1^n u_i \leq t, \ u_{i+1} \leq a_i \sum_1^i u_j; \quad i = 1, 2, \ldots, n-1\}$$

Making the change in variables $u_i = t\, t_i$, $i = 1, 2, \ldots, n$, we obtain

$$\mathcal{I}_n(t) = t^{\alpha_1 + \alpha_2 + \cdots + \alpha_n} \int\int_R \cdots \int f[t(t_1 + t_2 + \cdots + t_n)] t_1^{\alpha_1 - 1} t_2^{\alpha_2 - 1} \cdots t_n^{\alpha_n - 1} dt_1 dt_2 \cdots dt_n$$

where R is the same simplex as (3). Using (5) and after making an obvious change in variable we obtain

$$I_n(t) = [B_{\frac{a_1}{1+a_1}}(\alpha_2, \alpha_1)][B_{\frac{a_2}{1+a_2}}(\alpha_3, \alpha_2 + \alpha_1)] \cdots [B_{\frac{a_{n-1}}{1+a_{n-1}}}(\alpha_n, \alpha_{n-1} + \alpha_{n-2} + \ldots + \alpha_1)] \int_0^t f(\tau) \tau^{\alpha_1 + \alpha_2 + \ldots \alpha_n - 1} d\tau \tag{7}$$

(7) is of the same form as (5) except for the upper limit of the integral. We next consider the integral

$$K_n(t) = \int\int_T \cdots \int f[(\frac{v_1}{q_1})^{\beta_1} + (\frac{v_2}{q_2})^{\beta_2} + \cdots + (\frac{v_n}{q_n})^{\beta_n}] \cdot v_1^{\alpha_1 - 1} v_2^{\alpha_2 - 1} \cdots v_n^{\alpha_n - 1} dv_1 dv_2 \cdots dv_n \tag{8}$$

where T is the region of integration defined by the set of points

$$T = \{(v_1, v_2, \ldots, v_n) : v_i \geq 0, \sum_1^n \left(\frac{v_i}{q_i}\right)^{\beta_i} \leq t, \left(\frac{v_{i+1}}{q_{i+1}}\right)^{\beta_{i+1}}$$

$$\leq a_i \sum_1^i \left(\frac{v_j}{q_j}\right)^{\beta_j} ; i = 1, 2, \ldots, n\}$$

under the assumptions that $\beta_i > 0$, $\alpha_i > 0$, $q_i > 0$ for $i = 1, 2, \ldots, n$. Here, making the change in variable $\left(\frac{v_i}{q_i}\right)^{\beta_i} = u_i$, $i = 1, 2, \ldots, n$, we obtain

$$K_n(t) = \frac{q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_n^{\alpha_n}}{\beta_1\beta_2 \cdots \beta_n} \int\int_S \cdots \int f(u_1 + u_2 + \cdots + u_n)$$

$$u_1^{\frac{\alpha_1}{\beta_1}-1} u_2^{\frac{\alpha_2}{\beta_2}-1} \cdots u_n^{\frac{\alpha_n}{\beta_n}} du_1 du_2 \cdots du_n$$

which is of the same form as (6). Hence, using (7) we obtain

$$K_n(t) = \frac{q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_n^{\alpha_n}}{\beta_1\beta_2 \cdots \beta_n} [B_{\frac{a_1}{1+a_1}} \left(\frac{\alpha_2}{\beta_2}, \frac{\alpha_1}{\beta_1}\right)][B_{\frac{a_2}{1+a_2}} \left(\frac{\alpha_3}{\beta_3}, \frac{\alpha_2}{\beta_2} + \frac{\alpha_1}{\beta_1}\right)]$$

$$\cdots [B_{\frac{a_{n-1}}{1+a_{n-1}}} \left(\frac{\alpha_n}{\beta_n}, \frac{\alpha_{n-1}}{\beta_{n-1}} + \frac{\alpha_{n-2}}{B_{n-2}} + \cdots + \frac{\alpha_1}{\beta_1}\right)]$$

$$\int_0^t f(\tau)\tau^{\frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2}+\cdots+\frac{\alpha_n}{\beta_n}-1} d\tau \qquad (9)$$

4. Extensions

We now consider for r = 1, 2, ..., n and $0 \leq s \leq n - r$, the integral

$$J_{r,s} = \iint_R \cdots \int f(t_r + t_{+1} + \cdots + t_{r+s}) t_1^{\alpha_1 - 1} t_2^{\alpha_2 - 1} \cdots t_n^{\alpha_n - 1} dt_1 \, dt_2 \cdots dt_n$$

Where R is the same region as described in (3). The function $f(\cdot)$ and the parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$ are as defined before. It is possible to reduce $J_{1,n-2}$ and $J_{n,0}$ to single integrals although a simple reduction does not appear to exist for other values of r and s

a. The Integral $J_{1,n-2}$

We first show that a double integral of the form

$$H(a, b) = \int_0^b \int_0^a f[\tau(1 - v)A][\tau(1 - v)]^{\alpha-1} (\tau v)^{\beta-1} \tau \, d\tau \, dv$$

where $0 \leq a,b \leq 1$; $A > 0$; $\alpha,\beta > 0$ is reducible to single integrals. Making the change in variable $\tau(1 - v) = \theta$ and $(1 - v) = u$, we get

$$H(a, b) = \int_{1-b}^1 \int_0^{au} f(A\theta)\theta^{\alpha+\beta-1} (1 - u)^{\beta-1} u^{-\beta-1} d\theta \, du$$

Interchanging the order of integration yields

$$H(a, b) = \int_0^a f(A\theta)\theta^{\alpha+\beta-1} \int_{\frac{\theta}{a}}^1 (1 - u)^{\beta-1} u^{-\beta-1} du \, d\theta$$

$$- \int_0^{a(1-b)} f(A\theta)\theta^{\alpha+\beta-1} \int_{\frac{\theta}{a}}^{1-b} (1 - u)^{\beta-1} u^{-\beta-1} du \, d\theta$$

Making the change in variable $t = (1 - u)/u$ we obtain finally

$$H(a, b) = \frac{1}{\beta} \int_0^a f(A\theta)\theta^{\alpha+\beta-1} (\frac{a}{\theta} - 1)^\beta \, d\theta$$

$$- \frac{1}{\beta} \int_0^{a(1-b)} f(A\theta)\theta^{\alpha+\beta-1} [(\frac{a}{\theta} - 1)^\beta - (\frac{b}{1 - b})^\beta] d\theta$$

Now

$$\mathcal{J}_{1,n-2} = \int\int_R \cdots \int f(t_1 + t_2 + \cdots + t_{n-1}) t_1^{\alpha_1 - 1} t_2^{\alpha_2 - 1} \cdots t_n^{\alpha_n - 1}$$
$$dt_1 \; dt_2 \cdots dt_n$$

To evaluate $\mathcal{J}_{1,n-2}$ we perform the same sequence of variable transformation as in the evaluation of $\mathcal{I}_n$, and obtain

$$J_{1,n-2} = [B_{\frac{a_1}{1+a_1}}(\alpha_2, \alpha_1)][B_{\frac{a_2}{1+a_2}}(\alpha_3, \alpha_2 + \alpha_1)] \cdots [B_{\frac{a_{n-2}}{1+a_{n-2}}}(\alpha_{n-1}, \alpha_{n-2} + \alpha_{n-3} + \cdots + \alpha_1)]$$

$$\int_0^{\frac{a_{n-1}}{1+a_{n-1}}} \int_0^1 f[\tau(1 - u_{n-1})][\tau(1 - u_{n-1})]^{\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} - 1} (\tau\, u_{n-1})^{\alpha_n - 1} \tau \, d\tau \, du_{n-1}$$

This double integral is of the same form as $H(a, b)$ with $A = 1$, $a = 1$, $b = a_{n-1}/(1 + a_{n-1})$, $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}$, $\beta = \alpha_n$. *Hence*

$$\mathcal{J}_{1,n-2} = [B_{\frac{a_1}{1+a_1}}(\alpha_2, \alpha_1)][B_{\frac{a_2}{1+a_2}}(\alpha_3, \alpha_2 + \alpha_1)] \cdots [B_{\frac{a_{n-2}}{1+a_{n-2}}}(\alpha_{n-1}, \alpha_{n-2} + \alpha_{n-3} + \cdots + \alpha_1)]$$

$$\{ \frac{1}{\alpha_n} \int_0^1 f(\theta)\theta^{\alpha_1 + \alpha_2 + \cdots + \alpha_n - 1} (\frac{1}{\theta} - 1)^{\alpha_n} d\theta$$

$$- \frac{1}{\alpha_n} \int_0^{\frac{1}{1+a_{n-1}}} f(\theta)\theta^{\alpha_1 + \alpha_2 + \cdots + \alpha_n - 1} [(\frac{1}{\theta} - 1)^{\alpha_n} - (a_{n-1})^{\alpha_n}] d\theta$$

and in final form

$$\mathcal{J}_{1,n-2} = [B_{\frac{a_1}{1+a_1}}(\alpha_2, \alpha_1)][B_{\frac{a_2}{1+a_2}}(\alpha_3, \alpha_2 + \alpha_1)] \cdots [B_{\frac{a_{n-2}}{1+a_{n-2}}}(\alpha_{n-1}, \alpha_{n-2} + \alpha_{n-3} + \alpha_{n-3} + \cdots + \alpha_1)]$$

$$\{[\int_0^1 - \int_0^{\frac{1}{1+a_{n-1}}}] f(\theta)\theta^{\alpha_1+\alpha_2+\cdots+\alpha_{n-1}-1}(1-\theta)^{\alpha_n}\, d\theta$$

$$+ \frac{(a_{n-1})^{\alpha_n}}{\alpha_n}\int_0^{\frac{1}{1+a_{n-1}}} f(\theta)\theta^{\alpha_1+\alpha_2+\cdots+\alpha_n-1}\, d\theta\}$$

b. <u>The Integral $\mathcal{J}_{n,0}$</u>

We consider here a double integral of the form

$$\tilde{K}(a, b) = \int_0^b \int_0^a f(\tau vA)[\tau(1-v)]^{\alpha-1}(\tau v)^{\beta-1}\,\tau\, d\tau\, dv$$

where $0 \leq a_1 b \leq 1$; $A > 0$; $\alpha, \beta > 0$. Making the change in variable $\tau v = \theta$ and $v = u$, we get

$$\tilde{K}(a, b) = \int_0^b \int_0^{au} f(A\theta)\theta^{\alpha+\beta-1}(1-u)^{\alpha-1}\, u^{-\alpha-1}\, d\theta\, du$$

$$= \int_0^{ab} f(A\theta)\theta^{\alpha+\beta-1} \int_{\frac{\theta}{a}}^{b} (1-u)^{\alpha-1}u^{-\alpha-1}du\, d\theta$$

$$= \frac{1}{\alpha}\int_0^{ab} f(A\theta)\theta^{\alpha+\beta-1}[-(\frac{1}{b}-1)^{\alpha} + (\frac{a}{\theta}-1)^{\alpha}]d\theta$$

$$= -\frac{1}{\alpha}(\frac{1}{b}-1)^{\alpha}\int_0^{ab} f(A\theta)\theta^{\alpha+\beta-1}\, d\theta + \frac{1}{\alpha}\int_0^{ab} f(A\theta)\theta^{\beta-1}(a-\theta)^{\alpha}\, d\theta$$

Now

$$\mathcal{J}_{n,0} = \int\int_R \cdots \int f(t_n)t_1^{\alpha_1-1}\, t_2^{\alpha_2-1} \cdots t_n^{\alpha_n-1}\, dt_1\, dt_2 \cdots dt_n$$

Again, we perform the same sequence of variable transformation as in the evaluation of $\mathcal{I}_n$ and we obtain

$$\mathcal{J}_{n,0} = [B_{\frac{a_1}{1+a_1}}(\alpha_2, \alpha_1)][B_{\frac{a_2}{1+a_2}}(\alpha_3, \alpha_2+\alpha_1)] \cdots [B_{\frac{a_{n-2}}{1+a_{n-2}}}$$

$$(\alpha_{n-1}, \alpha_{n-2}+\alpha_{n-3}+\cdots+\alpha_1)] \int_0^{\frac{a_{n-1}}{1+a_{n-1}}}\int_0^1 f(\tau u_{n-1})$$

$$[\tau(1-u_{n-1})]^{\alpha_1+\alpha_2+\cdots+\alpha_{n-1}-1}(\tau u_{n-1})^{\alpha_n-1}\,\tau\, d\tau\, du_{n-1}$$

This double integral is of the same form as $\tilde{K}(a, b)$ with $A = 1$, $a = 1$, $b = a_{n-1}/(1 + a_{n-1})$, $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}$ and $\beta = \alpha_n$. Hence

$$\mathcal{J}_{n,0} = [B_{\frac{a_1}{1+a_1}}(\alpha_2, \alpha_1)][B_{\frac{a_2}{1+a_2}}(\alpha_3, \alpha_2 + \alpha_1)] \cdots [B_{\frac{a_{n-2}}{1+a_{n-2}}}(\alpha_{n-1}, \alpha_{n-2} + \alpha_{n-3} + \cdots + \alpha_1)].$$

$$\frac{1}{\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}} \left[\int_0^{\frac{a_{n-1}}{1+a_{n-1}}} f(\theta)\theta^{\alpha_n - 1}(1 - \theta)^{\alpha_1+\alpha_2+\cdots+\alpha_{n-1}} d\theta \right.$$

$$\left. - \left(\frac{1}{a_{n-1}}\right)^{\alpha_1+\alpha_2+\cdots+\alpha_{n-1}} \int_0^{\frac{a_{n-1}}{1+a_{n-1}}} f(\theta)\theta^{\alpha_1+\alpha_2+\cdots+\alpha_n - 1} d\theta \right]$$

5. A Still Further Extension.

Another multiple integral which is reducible to a single integral is the following

$$\iint_R \cdots \int f(t_1 + t_2 + \cdots + t_n)\, t_1^{\alpha_1 - 1} t_2^{\alpha_2 - 1} \cdots t_n^{\alpha_n - 1} \cdot (t_1 + t_2)^{\beta_2}(t_1 + t_2 + t_3)^{\beta_3} \cdots (t_1 + t_2 + \cdots t_{n-1})^{\beta_{n-1}}\, dt_1\, dt_2 \cdots dt_n$$

where $\beta_i \geq 0$, $(i = 2, 3, \cdots, n - 1)$. The reduction process is similar to that used for (4) and will not be repeated here.

6. Applications to Statistics

Statistical distributions of the multivariate type classified by functional form have received little attention. For univariate classes, the Pearson's type (e.g. see [6]) are well known. For the multivariate case, Lord [14] and Box and Hunter [3] consider the radical or spherical distributions where the joint probability density functions of the sequence of random variables $\{X_1, X_2, \cdots, X_n\}$ has the form

$$\phi_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = A\, f(x_1^2 + x_2^2 + \cdots + x_n^2)$$

where A is a constant. Kelker [11] studies linear transformations of spherically distributed variables, while Baldessari [2] considers density functions of the form

$$\phi_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = A\, f(x_1 + x_2 + \cdots + x_n)$$

where $\sum_{j=1}^{j=n} (x_j - a)^2 \leq r^2$, and of the form

$$\phi_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = A\, f[(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_n - a_n)^2]$$

where $\sum_{j=1}^{j=n} (x_j - a_j)^2 \leq r^2$. Finally, the so-called elliptically symmetric distributions are studied by McGraw and Wagner [15].

Power transformations of the type $X = Y^\lambda$ have been used in Dirichlet's distributions (see e.g. Johnston and Kotz [9]) and in the form $X = aY^\lambda - (1 - Y)^\lambda$ are known as the Tukey lambda transformation [8][10]. Generalizations to the standard Dirichlet distribution were considered by Johnston and Kotz [9].

We consider presently some generalizations of the previously investigated classes of statistical distributions. In particular, we study some properties of a class of multivariate distributions defined by a functional form in

which the variables may be subject to a power transformation.

Let then A be a positive constant and f(x) be a continuous nonnegative function defined everywhere in the interval $0 \le x < \infty$ with the property that for $\gamma_i > 0$, $i = 1, 2, \ldots, n$, the improper integral

$$\int_0^\infty f(\tau)\tau^{\gamma_1+\gamma_2+\cdots+\gamma_n-1}\, d\tau$$

exists and converges absolutely.

Let $\{X_1, X_2, \ldots, X_n\}$ be an n-dimensional non-negative random variable whose P.D.F. $\phi_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n)$ is equal to

$$A\, f[(\frac{x_1}{q_1})^{\beta_1} + (\frac{x_2}{q_2})^{\beta_2} + \cdots + (\frac{x_n}{q_n})^{\beta_n}]x_1^{\alpha_1-1}x_2^{\alpha_2-1}\cdots x_n^{\alpha_n-1} \tag{10}$$

over the region $\{(x_1, x_2, \ldots, x_n) : 0 \le x_i < \infty, i = 1, 2, \ldots, n\}$, and is zero elsewhere, and let for $i = 1, 2, \ldots, n$ the quantities $\alpha_i > 0$, $\beta_i > 0$, $q_i > 0$ be uniquely defined.

a. <u>*The Constant A*</u>

The quantity A is given by the normalizing conditions

$$\frac{1}{A} = \int_0^\infty\int_0^\infty\cdots\int_0^\infty f[(\frac{x_1}{q_1})^{\beta_1} + (\frac{x_2}{q_2})^{\beta_2} + \cdots + (\frac{x_n}{q_n})^{\beta_n}]$$
$$x_1^{\alpha_1-1}x_2^{\alpha_2-1}\cdots x_n^{\alpha_n-1}\, dx_1dx_2\cdots dx_n$$

The multiple integral is of the same form as (8) with $t = \infty$ and $a_i = \infty$, $i = 1, 2, \ldots, n$. Hence

$$\frac{1}{A} = \frac{q_1^{\alpha_1}q_2^{\alpha_2}\cdots q_n^{\alpha_n}}{\beta_1\beta_2\cdots\beta_n}\ \frac{\Gamma(\frac{\alpha_1}{\beta_1})\Gamma(\frac{\alpha_2}{\beta_2})\cdots\Gamma(\frac{\alpha_n}{\beta_n})}{\Gamma(\frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2}+\cdots+\frac{\alpha_n}{\beta_n})}\ .$$
$$\int_0^\infty f(\tau)\tau^{\frac{\alpha_1}{\beta_1}+\frac{\alpha_2}{\beta_2}+\cdots+\frac{\alpha_n}{\beta_n}-1}\, d\tau \tag{11}$$

b. Mixed Moments

The mixed moments about the origin if they exist, are defined for $r_i = 1, 2, \ldots, (i = 1, 2, \ldots, n)$ by the expression

$$\mu'_{r_1 r_2 \ldots r_n} = \int_0^\infty \int_0^\infty \cdots \int_0^\infty x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n} \phi_{X_1, X_2, \ldots X_n}(x_1, x_2, \ldots, x_n) \, dx_1, dx_2 \cdots dx_n$$

$$= A \frac{q_1^{\alpha_1 + r_1} q_2^{\alpha_2 + r_2} \cdots q_n^{\alpha_n + r_n}}{\beta_1 \beta_2 \cdots \beta_n} \frac{\Gamma(\frac{\alpha_1 + r_1}{\beta_1}) \Gamma(\frac{\alpha_2 + r_2}{\beta_2}) \cdots \Gamma(\frac{\alpha_n + r_n}{\beta_n})}{\Gamma(\frac{\alpha_1 + r_1}{\beta_1} + \frac{\alpha_2 + r_2}{\beta_2} + \cdots + \frac{\alpha_n + r_n}{\beta_n})}$$

$$\int_0^\infty f(\tau) \tau^{\frac{\alpha_1 + r_1}{\beta_1} + \frac{\alpha_2 + r_2}{\beta_2} + \cdots + \frac{\alpha_n + r_n}{\beta_n} - 1} d\tau$$

c. Marginal Distributions

The marginal PDF $\phi_{X_i}(\cdot)$ of X_i is given by $(0 \leq x_i < \beta)$

$$\phi_{X_i}(x_i) = \int_0^\infty \int_0^\infty \cdots \int_0^\infty \phi_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) \, dx_1 dx_2 \ldots dx_{i-1} dx_{i+1} \ldots dx_n$$

$$= A \frac{q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_{i-1}^{\alpha_{i-1}} q_{i+1}^{\alpha_{i+1}} \cdots q_n^{\alpha_n}}{\beta_1 \beta_2 \cdots \beta_{i-1} \beta_{i+1} \cdots \beta_n} .$$

$$x_i^{\alpha_i - 1} \int_0^\infty f[\tau + (\frac{x_i}{q_i})^{\beta_i}] \, \tau^{\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + \frac{\alpha_{i-1}}{\beta_{i-1}} + \frac{\alpha_{i+1}}{\beta_{i+1}} + \cdots + \frac{\alpha_n}{\beta_n} - 1} d\tau$$

By a suitable change in the order of integration, setting $y = \tau + (x_i/q_i)^{\beta_i}$, and using the convolution theorem, one can verify that $\int_0^\infty \phi_{X_i}(x_i) = 1$. In a similar fashion, it is possible to show that the joint marginal P.D.F. of $\{X_1, X_2, \ldots, X_k\}$ $(k \leq n)$ is given by

$$\phi_{X_1, X_2, \ldots, X_k}(x_1, x_2, \ldots, x_k) = A \frac{q_{k+1}^{\alpha_{k+1}} q_{k+2}^{\alpha_{k+2}} \cdots q_n^{\alpha_n}}{\beta_{k+1}\beta_{k+2} \cdots \beta_n} x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} \cdots x_k^{\alpha_k - 1}$$

$$\int_0^\infty f[(\frac{x_1}{q_1})^{\beta_1} + (\frac{x_2}{q_2})^{\beta_2} + \cdots + (\frac{x_n}{q_n})^{\beta_n} + \tau] \, \tau^{\frac{\alpha_{k+1}}{\beta_{k+1}} + \frac{\alpha_{k+2}}{\beta_{k+2}} + \cdots + \frac{\alpha_n}{\beta_n} - 1} d\tau$$

The conditional distributions can then readily be obtained. We presently extablish the following two theorems

d. Theorem 1

The $(n - 1)$ random variables $\{Y_1, Y_2, \ldots, Y_{n-1}\}$ where

$$Y_i = \frac{(\frac{X_i}{q_i})^{\beta_i}}{(\frac{X_1}{q_1})^{\beta_1} + (\frac{X_2}{q_2})^{\beta_2} + \cdots + (\frac{X_n}{q_n})^{\beta_n}}, \quad i = 1, 2, \ldots, n - 1$$

have a joint probability density function which is of the Dirichlet type inside the simplex

$$\sum = \{(y_1, y_2, \ldots, y_{n-1}) : y_1 + y_2 + \cdots + y_{n-1} \le 1, y_1 \ge 0, y_2 \ge 0, \ldots, y_{n-1} \ge 0\}$$

and which is zero elsewhere.

Proof:

Let $Y_n = (\frac{X_1}{q_1})^{\beta_1} + \cdots + (\frac{X_n}{q_n})^{\beta_n}$

Then $(\frac{X_i}{q_i})^{\beta_i} = Y_i Y_n, \quad i = 1, 2, \ldots, n - 1$

and $(\frac{X_n}{q_n})^{\beta_n} = (1 - Y_1 - Y_2 - \cdots - Y_{n-1}) Y_n$

This particular transformation maps the region $\{(x_1, x_2, \ldots, x_n) : 0 \leq x_i < \infty,$ $i = 1, 2, \ldots, n\}$ onto the region $C = \{(y_1, y_2, \ldots, y_n) : y_1 + y_2 + \cdots + y_{n-1} \leq 1,$ $y_i \geq 0$ for $i = 1, 2, \ldots, n - 1,\ 0 \leq y_n < \infty\}$. The corresponding inverse functions for this one to one transformation is

$$X_i = q_i(Y_iY_n)^{1/\beta_i}, \qquad i = 1, 2, \ldots, n - 1$$

$$X_n = q_n[(1 - Y_1 - Y_2 - \cdots Y_{n-1})Y_n]^{1/\beta_n}$$

The Jacobian of the transformation is

$$J = \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(y_1, y_2, \ldots, y_n)}$$

$$= \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$

Now for $i = 1, 2, \ldots, n$, we have

$$\frac{\partial x_i}{\partial y_i} = \frac{q_i}{\beta_i}(y_iy_n)^{\frac{1}{\beta_i} - 1} y_n$$

$$\frac{\partial x_i}{\partial y_n} = \frac{q_i}{\beta_i}(y_iy_n)^{\frac{1}{\beta_i} - 1} y_i$$

$$\frac{\partial x_i}{\partial y_j} = 0 \quad \text{for } j \neq i, n$$

Also for $i = 1, 2, \ldots, n - 1$

$$\frac{\partial x_n}{\partial y_i} = \frac{q_n}{\beta_n}(-y_n)[(1 - y_1 - y_2 - \cdots - y_{n-1})y_n]^{\frac{1}{\beta_n} - 1}$$

and $$\frac{\partial x_n}{\partial y_n} = \frac{q_n}{\beta_n}(1 - y_1 - y_2 - \cdots - y_{n-1})[(1 - y_1 - y_2 - \cdots - y_{n-1})y_n]^{\frac{1}{\beta_n} - 1}$$

The value of the Jacobian is thus

$$J = \frac{q_1}{\beta_1}(y_1 y_n)^{\frac{1}{\beta_1} - 1} \cdot \frac{q_2}{\beta_2}(y_2 y_n)^{\frac{1}{\beta_2} - 1} \cdots \frac{q_{n-1}}{\beta_{n-1}}(y_{n-1} y_n)^{\frac{1}{\beta_{n-1}} - 1} \cdot$$

$$\frac{q_n}{\beta_n}[(1 - y_1 - y_2 - \cdots - y_{n-1})y_n]^{\frac{1}{\beta_n} - 1} \cdot$$

$$\begin{vmatrix} y_n & 0 & \cdots & y_1 \\ 0 & y_n & \cdots & y_2 \\ \vdots & \vdots & \cdots & \vdots \\ -y_n & -y_n & \cdots & (1-y_1-y_2-\cdots-y_{n-1}) \end{vmatrix}$$

Noting that the value of the determinant is y_n^{n-1}, we have

$$J = \frac{q_1 q_2 \cdots q_n}{\beta_1 \beta_2 \cdots \beta_n} y_1^{\frac{1}{\beta_1} - 1} y_2^{\frac{1}{\beta_2} - 1} \cdots y_{n-1}^{\frac{1}{\beta_{n-1}} - 1} y_n^{\frac{1}{\beta_1} + \frac{1}{\beta_2} + \cdots \frac{1}{\beta_n} - 1}$$

$$(1 - y_1 - y_2 - \cdots - y_{n-1})^{\frac{1}{\beta_n} - 1}$$

Hence, the joint P.D.F. of $\{Y_1, Y_2, \ldots, Y_{n-1}, Y_n\}$ is for $(y_1, y_2, \ldots, y_n) \in C$

$$A \;\; f(y_n) q_1^{\alpha_1 - 1}(y_1 y_n)^{\frac{\alpha_1 - 1}{\beta_1}} \cdot q_2^{\alpha_2 - 1}(y_2 y_n)^{\frac{\alpha_2 - 1}{\beta_2}} \cdots q_{n-1}^{\alpha_{n-1} - 1}(y_{n-1} y_n)^{\frac{\alpha_{n-1} - 1}{\beta_{n-1}}}$$

$$q_n^{\alpha_n - 1}[(1 - y_1 - y_2 - \cdots - y_{n-1})y_n]^{\frac{\alpha_n - 1}{\beta_n}} \cdot |J|$$

$$= A \frac{q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_n^{\alpha_n}}{\beta_1 \beta_2 \cdots \beta_n} y_1^{\frac{\alpha_1}{\beta_1}-1} y_2^{\frac{\alpha_2}{\beta_2}-1} \cdots y_{n-1}^{\frac{\alpha_{n-1}}{\beta_{n-1}}} \cdot (1 - y_1 - y_2 - \cdots - y_{n-1})^{\frac{\alpha_n}{\beta_n}-1}$$

$$\cdot f(y_n) y_n^{\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + \cdots + \frac{\alpha_n}{\beta n} - 1}$$

and is zero elsewhere. Using the computed value of the constant A in (11) we obtain for the joint P.D.F. of $\{Y_1, Y_2, \ldots, Y_{n-1}\}$ inside the simplex $\sum$ to be

$$\frac{\Gamma(\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + \cdots + \frac{\alpha_n}{\beta_n})}{\Gamma(\frac{\alpha_1}{\beta_1})\, \Gamma\, (\frac{\alpha_2}{\beta_2}) \cdots \Gamma(\frac{\alpha_n}{\beta_n})} y_1^{\frac{\alpha_1}{\beta_1}-1} y_2^{\frac{\alpha_2}{\beta_2}-1} \cdots y^{\frac{\alpha_n}{\beta_{n-1}}-1}$$

$$(1 - y_1 - y_2 - \cdots - y_{n-1})^{\frac{\alpha_n}{\beta_n}-1}$$

and to be zero elsewhere.

e. Theorem 2

The n random variables $\{Y_1, Y_2, \ldots, Y_n\}$ where

$$Y_i = \frac{\left(\frac{X_{i+1}}{q_{i+1}}\right)^{\beta_{i+1}}}{\left(\frac{X_1}{q_1}\right)^{\beta_1} + \left(\frac{X_2}{q_2}\right)^{\beta_2} + \cdots + \left(\frac{X_i}{q_i}\right)^{\beta_i}} \qquad i = 1, 2, \ldots, n-1$$

$$Y_n = \left(\frac{X_1}{q_1}\right)^{\beta_1} + \left(\frac{X_2}{q_2}\right)^{\beta_2} + \cdots + \left(\frac{X_n}{q_n}\right)^{\beta_n}$$

are independently distributed. The marginal P.D.F. of Y_i's are

$$\phi_{Y_i}(y_i) = \begin{cases} A_i \dfrac{y_i^{\frac{\alpha_{i+1}}{\beta_{i+1}} - 1}}{(1 + y_i)^{\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + \cdots + \frac{\alpha_{i+1}}{\beta_{i+1}}}} & , \quad 0 < y_i < \infty \quad (i = 1, 2, \ldots, n-1) \\ 0 & , \quad \text{otherwise} \end{cases}$$

$$\phi_{Y_n}(y_n) = \begin{cases} A_n f(y_n) y_n^{\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + \cdots + \frac{\alpha_n}{\beta_n} - 1} & , \quad 0 < y_n < \infty \\ 0 & , \quad \text{otherwise} \end{cases}$$

where the A_i's are normalizing constants.

Proof:

The joint distribution function of $\{Y_1, Y_2, \ldots, Y_n\}$ is given by

$P\{Y_1 \leq y_1, Y_2 \leq y_2, \ldots, Y_n \leq y_n\} =$

$$P\left\{\left(\frac{X_2}{q_2}\right)^{\beta_2} \leq y_1 \left(\frac{X_1}{q_1}\right)^{\beta_1}, \left(\frac{X_3}{q_3}\right)^{\beta_3} \leq y_2 \left[\left(\frac{X_1}{q_1}\right)^{\beta_1} + \left(\frac{X_2}{q_2}\right)^{\beta_2}\right], \ldots,\right.$$

$$\left.\left(\frac{X_n}{q_n}\right)^{\beta_n} \leq y_{n-1} \sum_{i=1}^{i=n-1} \left(\frac{X_i}{q_i}\right)^{\beta_i}, \sum_{i=1}^{i=n} \left(\frac{X_i}{q_i}\right)^{\beta_i} \leq y_n\right\}$$

$$= A \int\int_{\tilde{T}}\cdots\int f\left[\left(\frac{x_1}{q_1}\right)^{\beta_1} + \left(\frac{x_2}{q_2}\right)^{\beta_2} + \cdots + \left(\frac{x_n}{q_n}\right)^{\beta_n}\right]$$

$$x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} \cdots x_n^{\alpha_n - 1}\, dx_1\, dx_2 \cdots dx_n$$

where $\tilde{T}$ is the region of integration defined by the set of points

$$\tilde{T} = \{(x_1, x_2, \ldots, x_n) : x_i \geq 0, \sum_1^n \left(\frac{x_i}{q_i}\right)^{\beta_i} \leq y_n, \left(\frac{x_{i+1}}{q_{i+1}}\right)^{\beta_{i+1}}$$

$$\leq y_i \sum_1^n \left(\frac{x_j}{q_j}\right)^{\beta_i} ; i = 1, 2, \ldots, n\}$$

Thus, from (9)

$$P\{Y_1 \leq y_1, Y_2 \leq y_2, \ldots, Y_n \leq y_n\} = A \frac{q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_n^{\alpha_n}}{\beta_1\beta_2 \cdots \beta_n}$$

$$\left[B_{\frac{y_1}{1+y_1}}\left(\frac{\alpha_2}{\beta_2}, \frac{\alpha_1}{\beta_1}\right)\right] \cdot$$

$$\left[B_{\frac{y_2}{1+y_2}}\left(\frac{\alpha_3}{\beta_3}, \frac{\alpha_2}{\beta_2} + \frac{\alpha_1}{\beta_1}\right)\right] \cdots \left[B_{\frac{y_{n-1}}{1+y_{n-1}}}\left(\frac{\alpha_n}{\beta_n}, \frac{\alpha_{n-1}}{\beta_{n-1}} + \frac{\alpha_{n-2}}{\beta_{n-2}} + \cdots + \frac{\alpha_1}{\beta_1}\right)\right] \cdot$$

$$\int_0^{y_n} f(\tau)\tau^{\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + \cdots + \frac{\alpha_n}{\beta_n} - 1}\, dy_n$$

which establishes the independence of $Y_1, Y_2, \ldots, Y_n$. The distribution function of Y_i is $(0 \leq y_i < \infty)(i = 1, 2, \ldots, n - 1)$

$$A_i \; B_{\frac{y_i}{1+y_i}}\left(\frac{\alpha_{i+1}}{\beta_{i+1}}, \frac{\alpha_i}{\beta_i} + \frac{\alpha_{i-1}}{\beta_{i-1}} + \cdots + \frac{\alpha_1}{\beta_1}\right)$$

$$= A_i \int_0^{\frac{y_i}{1+y_i}} y^{\frac{\alpha_{i+1}}{\beta_{i+1}} - 1} (1 - y)^{\frac{\alpha_i}{\beta_i} + \frac{\alpha_{i-1}}{\beta_{i-1}} + \cdots + \frac{\alpha_1}{\beta_1} - 1}\, dy$$

where $$A_i = \frac{1}{B\left(\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + \cdots + \frac{\alpha_i}{\beta_i}, \frac{\alpha_{i+1}}{\beta_{i+1}}\right)}$$

Hence the P.D.F. of Y_i $(i = 1, 2, \ldots, n-1)$ is for $0 < y_i < \infty$

$$\phi_{Y_i}(y_i) = A_i \frac{y_i^{\frac{\alpha_{i+1}}{\beta_{i+1}} - 1}}{(1 + y_i)^{\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + \cdots + \frac{\alpha_{i+1}}{\beta_{i+1}}}}$$

which can be recognized as a beta distribution of the second kind. The P.D.F. of Y_n is $(0 \leq y_n < \infty)$

$$\phi_{Y_n}(y_n) = A_n\, f(y_n)\, y_n^{\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + \cdots + \frac{\alpha_n}{\beta_n} - 1}$$

where $$\frac{1}{A_n} = \int_0^\infty f(\tau)\tau^{\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + \cdots + \frac{\alpha_n}{\beta_n} - 1} d\tau$$

Theorems 1 and 2 are generalizations of some well known results in statistics (see e.g. Wilks [19] and Aitchison [1]).

References

[1] J. Aitchison, Inverse Distributions and Independent Gamma-Distributed Products of Random Variables, Biometrika, 50 (1963) pp. 505-509.

[2] B. Baldessari, Osservazioni Sulle Funsioni Caratteristche Di Alcune Classi Di V. C. n-dimensionali. Publicazioni della Seconda Serie dell' Instituto di Calcolo delle Probabilita dell' Universita di Roma, 49 (1965) pp. 1-11.

[3] G. E. P. Box and J. S. Hunter, Multi-Factor Experimental Designs for Exploring Response Surfaces, Ann. Math Stat. 28 (1957); pp. 195-241.

[4] J. L. Coolidge, A History of Geometrical Methods, Oxford University Press, London, 1040.

[5] J. Edwards, A Treatise on the Integral Calculus, Vol. 2. Macmillan, N. Y., 1922.

[6] W. P. Elderton, Frequency Curves and Correlation, 3rd ed. Cambridge, 1938.

[7] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series and Products, Academic Press, N.Y., 1965.

[8] C. Hastings, F. Mosteller, J. W. Tukey and C. P. Winsor, Low Moments for Small Samples: A Comparative Study of Order Statistics, Ann. Math. Stat., 18 (1947); pp. 413-426.

[9] N. L. Johnson and S. Kotz, Distributions in Statistics: Continuous Multivariate Distributions, J. Wiley & Sons, N. Y., 1972.

[10] B. L. Joiner and J. R. Rosenblatt, Some Properties of Range in Samples from Tukey's Symmetric Lambda Distributions, Jour. Amer. Stat. Assn., 66 (1971) pp. 992-995.

[11] D. Kelker, Distribution Theory of Spherical Distributions and a Location-Scale Parameter Generalization, Sankhya, Series A, 32 (1970) pp. 419-430.

[12] M. G. Kendall and A. Stuart, The Advanced Theory of Statistics, Vol. 1, Charles Griffi & Co. Ltd., London, 1969.

[13] M. S. Klamkin, Extensions of Dirichlet's Multiple Integral, SIAM J. Math. Anal. 2 (1971) pp. 467-469.

[14] R. D. Lord, The Use of Hankel Transforms in Statistics, I, General Theory and Examples, Biometrika, 41 (1954); pp. 44-55.

[15] D. K. McGraw and J. F. Wagner, Elliptically Symmetric Distributions, IEEE Transactions on Information Theory, 14 (1968), pp. 110-120.

[16] B. D. Sivazlian, The Generalized Dirichlet's Multiple Integral, SIAM Review, 11 (1969) pp. 285-288.

[17] ________________, A Class of Multiple Integrals, SIAM J. Math. Anal. 2 (1971) pp. 72-75.

[18] E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, Cambridge University Press, London, 1927.

[19] S. S. Wilks, Mathematical Statistics, J. Wiley & Sons, N. Y., 1962.